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Albert-Ludwigs-Universität Freiburg
Sommersemester 2024

Topology

Problem Sheet 5

Deadline: 28 May 2024, 15h

Exercise 1 (4 Points).

Let $(X, \mathcal{T})$ be a topological space.

a) Show that X is locally connected if and only if for every open subset U of X, each connected component of U (with respect to the subspace topology) is open in X.

b) Show that every open subset of X has countably many connected components if X is locally connected and second-countable.

c) How many connected components does $\mathbb{R} \setminus \mathbb{Q}$ equipped with the subspace topology from $\mathbb{R}$ with the euclidean topology? Is the topological space $\mathbb{R} \setminus \mathbb{Q}$ second-countable?

Exercise 2 (4 Points).

a) Show that $\mathcal{T} = \{\emptyset\} \cup \{U \subset \mathbb{R} \,|\, 0 \in U\}$ defines a topologie on $\mathbb{R}$ such that $\{0\}$ is dense. Is this topological space T_1?

b) Is $(\mathbb{R}, \mathcal{T})$ connected?

c) Determine whether $(\mathbb{R}, \mathcal{T})$ is first or second-countable.

Exercise 3 (4 Points).

Let X be the metric space of all continuous function from $[0, 1]$ to $\mathbb{R}$ with the metric

$$\mathrm{d}_\infty(f, g) = \sup_{0 \le x \le 1} |f(x) - g(x)|.$$

Use the polynomials with coefficients in $\mathbb{Q}$ to determine whether X is separable.

Exercise 4 (8 Points).

Given a non-empty topological space $(X, \mathcal{T})$, let $\beta(X)$ be the collection of all ultrafilters on X.

a) Show that the family $([A])_{A \subset X}$ defines a basis of a topology on $\beta(X)$, where

$$[A] = \{\ \mathcal{U} \in \beta(X) \mid A \in \mathcal{U}\ \}.$$

Is this topology 0-dimensional? Is it Hausdorff?

b) Show that there is an injective map $f : \beta(X) \to \prod_{A \in \mathcal{P}(X)} \{0, 1\}$. If we equip $\{0, 1\}$ with the discrete topology and $\prod_{A \in \mathcal{P}(X)} \{0, 1\}$ with the product topology, is the image of $\beta(X)$ closed?

c) Show that the above injection is a homeomorphism between $\beta(X)$ and $\mathrm{Im}(f)$ equipped with the subspace topology.

d) Show that the collection of principal ultrafilters is dense in $\beta(X)$. If X is countable, is $\beta(X)$ separable?

Die Übungsblätter können zu zweit eingereicht werden. Abgabe der Übungsblätter im entsprechenden Fach im Keller des mathematischen Instituts.